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General triplectic quantization

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Abstract

The general structure of the $Sp(2)$ covariant version of the field-antifield quantization of general constrained systems in the Lagrangian formalism, the so called triplectic quantization, as presented in our previous paper with A.M.Semikhatov is further generalized and clarified. We present new unified expressions for the generating operators which are more invariant and which yield a natural realization of the operator V^a and provide for a geometrical explanation for its presence. This V^a operator provides then for an invariant definition of a degenerate Poisson bracket on the triplectic manifold being nondegenerate on a naturally defined submanifold. We also define inverses to nondegenerate antitriplectic metrics and give a natural generalization of the conventional calculus of exterior differential forms which *e.g.* explains the properties of these inverses. Finally we define and give a consistent treatment of second class hyperconstraints.

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1 Introduction

A general $Sp(2)$ -covariant Lagrangian quantization of general gauge theories was presented in [1] and was nick-named "triplectic quantization". The paper was based on the $Sp(2)$ -symmetric formalism developed in [2]–[5] which had its origin in a BRST-antiBRST Hamiltonian formalism. (For other works on the $Sp(2)$ formalism see also [6]–[14].) In [1] almost all general properties of the antisymplectic formalism was generalized to the $Sp(2)$ -case. In fact, even the antisymplectic formalism was slightly generalized.

The purpose of the present paper is to make some improvements, clarifications and further generalizations of the triplectic quantization as presented in [1]. Our main new results are new unified expressions for the generating operators Δ_{\pm}^a , expressions which are more invariant and which provide for a natural covariant realization of the V^a operator and a geometrical explanation for its presence in the $Sp(2)$ covariant formulation. We also show that the allowed form of V^a is more general than those considered in [1]. In fact, the natural arbitrariness of V^a is parametrized by a bosonic function which is arbitrary up to the requirement that it must satisfy the classical master equation. Different choices of V^a correspond then to different boundary conditions on the master action. Thus, we resolve the clash between the V^a in [2]–[4] and in [1],[5]. The interesting Poisson bracket introduced in [1], which is defined on a submanifold of the triplectic manifold, is generalized to include more general V^a operators. In fact, the unified expressions for Δ_{\pm}^a allow us to give a completely invariant definition of a degenerate Poisson bracket on the triplectic manifold which is nondegenerate on a naturally defined submanifold.

The nondegeneracy properties of the antitriplectic metric are here specified by the requirement of the existence of an inverse which at first glance look rather peculiar. However, by means of a generalized exterior differential form calculus, which we introduce, this inverse may be written as coefficients of a two-form in terms of which its properties become natural ones. The exterior differential is here an $Sp(2)$ vector which satisfies the same algebraic properties as the generating operators Δ_{\pm}^a . Our specification of an inverse to a nondegenerate antitriplectic metric allows us also to introduce second class hyperconstraints and to define the triplectic counterpart of the Dirac bracket.

The paper is organized as follows: In section 2 we give the general properties of the basic objects in triplectic quantization generalizing the results of [1] to general V^a operators. Inverses to triplectic metric and generalized differential forms are introduced. In section 3 we present our new unified expressions for the generating operators which are more invariant and which lead to natural realizations of V^a and explain their origin. In section 4 we point out how the gauge fixing procedure of [1] is generalized to more general V^a operators, and finally in section 5 we present the treatment of triplectic second class constraints. In an appendix we give some basic objects in terms of Darboux coordinates.

2 Basics of general triplectic quantization

In general triplectic quantization one considers a triplectic manifold \mathcal{M} with local coordinates Γ^A , $A = 1, \dots, 6N$, with Grassmann parities $\varepsilon(\Gamma^A) \equiv \varepsilon_A \in \{0, 1\}$. \mathcal{M} is endowed with a volume measure $d\mu(\Gamma) \equiv \rho(\Gamma)[d\Gamma]$ where $\rho(\Gamma)$ is a scalar density. On \mathcal{M} we have

the basic tensor $E^{ABa}(\Gamma)$, $a = 1, 2$, the antitriplectic metric, satisfying the properties

$$E^{ABa} = -E^{BAa}(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}, \quad \varepsilon(E^{ABa}) = \varepsilon_A + \varepsilon_B + 1. \quad (2.1)$$

It is required to be nondegenerate in the sense that there should exist a tensor Y_{bcBC}^a with the properties

$$\varepsilon(Y_{bcBC}^a) = \varepsilon_B + \varepsilon_C + 1, \quad Y_{bcBC}^a = -Y_{cbCB}^a(-1)^{\varepsilon_B\varepsilon_C}, \quad (2.2)$$

satisfying

$$E^{ABb}Y_{bcBC}^a = \delta_C^A \delta_c^a. \quad (2.3)$$

It follows then from (2.1), (2.2) and (2.3) that the conjugate equation holds as well, *i.e.*

$$Y_{cbCB}^a E^{BAb} = \delta_C^A \delta_c^a. \quad (2.4)$$

Notice, however, that Y_{bcBC}^a is not uniquely determined by these conditions.

In triplectic quantization there are two basic pairs of odd (fermionic) differential operators, Δ_+^a and Δ_-^a , which generate the master equations. Their dependence on \hbar is given by

$$\Delta_\pm^a \equiv \Delta^a \pm \frac{i}{\hbar} \hat{V}^a, \quad a = 1, 2, \quad (2.5)$$

where Δ^a are second order differential operators with respect to Γ^A , and \hat{V}^a first order ones. Δ^a and \hat{V}^a are defined by the expressions

$$\Delta^a = \frac{1}{2}(-1)^{\varepsilon_A} \rho^{-1} \partial_A \circ \rho E^{ABa} \partial_B, \quad a = 1, 2, \quad (2.6)$$

$$\begin{aligned} \hat{V}^a &\equiv V^a + \frac{1}{2} \operatorname{div} V^a, \quad a = 1, 2, \\ V^a &\equiv (-1)^{\varepsilon_A} V^{Aa} \partial_A, \quad \varepsilon(V^{Aa}) = \varepsilon_A + 1, \\ \operatorname{div} V^a &\equiv \rho^{-1} \left(\partial_A \rho V^{Aa} \right) (-1)^{\varepsilon_A}. \end{aligned} \quad (2.7)$$

The factor $(-1)^{\varepsilon_A}$ in the expression for V^a is inserted for convenience. Notice also that the antisymmetry properties (2.1) of E^{ABa} just provide for the required symmetry of the coefficients for $\partial_B \partial_A$ in (2.6).

Δ_\pm^a are hermitian operators with respect to the inner product

$$(f, g) \equiv \int f^*(\Gamma) g(\Gamma) d\mu(\Gamma) \quad (2.8)$$

for real coordinates, or provided one imposes the complex structure induced by the differential complex-conjugate conditions

$$(d\Gamma^A)^* = d\Gamma^B I_B^A, \quad \partial_B = I_B^A \partial_A^*, \quad (2.9)$$

where I_B^A are fields which satisfy the conditions

$$(I_B^A)^* I_C^B (-1)^{\varepsilon_C(1+\varepsilon_B)} = \delta_C^A \Rightarrow |\operatorname{sdet} I|^2 = 1, \quad (2.10)$$

$$\partial_C I_B^A - \partial_B I_C^A (-1)^{\varepsilon_B \varepsilon_C} = 0, \quad (2.11)$$

$$\partial_B^* I_A^B (-1)^{\varepsilon_B (\varepsilon_A + 1)} = \partial_A \ln \text{sdet } I = 0. \quad (2.12)$$

The last relation follows also from the requirement of a real measure ($\text{sdet } I = 1$). Under these conditions we have

$$(E^{CDa})^* = (-1)^{\varepsilon_A (\varepsilon_D + 1)} I_A^D E^{ABa} I_B^C, \quad (V^{Aa})^* = (-1)^{\varepsilon_B} V^{Ba} I_B^A. \quad (2.13)$$

Although we shall not make use of this complex structure in the following it should be useful to know that it may always be imposed.

The basic odd operators (2.5) are required to satisfy

$$[\Delta_\pm^a, \Delta_\pm^b] = 0 \quad \Leftrightarrow \quad \Delta_\pm^{\{a} \Delta_\pm^{b\}} = 0, \quad (2.14)$$

where the commutator $[\ , \]$ here and in the sequel denotes the graded supercommutator $[A, B] = AB - (-1)^{\varepsilon(A)\varepsilon(B)} BA$, and where the curly bracket indicates symmetrization in the indices a and b . These conditions imply in turn the following conditions on Δ^a and \hat{V}^a by identifications of different powers of i/\hbar in (2.14) (The conditions from Δ_+^a and Δ_-^a are identical.)

$$[\Delta^a, \Delta^b] = 0 \quad \Leftrightarrow \quad \Delta^{\{a} \Delta^{b\}} = 0, \quad (2.15)$$

$$[\Delta^{\{a}, \hat{V}^{b\}}] = 0 \Leftrightarrow \Delta^{\{a} \hat{V}^{b\}} + \hat{V}^{\{a} \Delta^{b\}} = 0, \quad (2.16)$$

and

$$[\hat{V}^a, \hat{V}^b] = 0 \quad \Leftrightarrow \quad \hat{V}^{\{a} \hat{V}^{b\}} = 0. \quad (2.17)$$

Identifying furthermore the coefficients for different powers of ∂_A we find that the conditions (2.15) imply

$$E^{AD\{a} \partial_D E^{BC|b\}} (-1)^{(\varepsilon_A + 1)(\varepsilon_C + 1)} + \text{cycle}(A B C) = 0, \quad (2.18)$$

$$(-1)^{\varepsilon_A} \rho^{-1} \partial_A \left(\rho E^{AB\{a} \partial_B (-1)^{\varepsilon_C} \rho^{-1} \partial_C (\rho E^{CD|b\}} \right) = 0. \quad (2.19)$$

They are conditions on E^{ABa} and ρ . One may notice that (2.18) imply the following conditions on Y_{bcBC}^a in (2.3):

$$\begin{aligned} & (-1)^{\varepsilon_C \varepsilon_A} E^{FBb} E^{GAa} \left(\delta_a^{\{f} \partial_A Y_{bcBC}^{g\}} (-1)^{\varepsilon_C \varepsilon_A} + \right. \\ & \left. + \text{cycle}[(a, A), (b, B), (c, C)] \right) E^{CHc} (-1)^{(\varepsilon_G + 1)(\varepsilon_H + \varepsilon_F + \varepsilon_B)} = 0. \end{aligned} \quad (2.20)$$

These conditions may easily be understood in terms of appropriately generalized differential forms. The basic objects for such a form calculus are the exterior differentials d^a , $a = 1, 2$, satisfying

$$[d^a, d^b] = 0 \quad \Leftrightarrow \quad d^{\{a} d^{b\}} = 0. \quad (2.21)$$

Notice the algebraic similarity with the properties (2.14) of Δ_{\pm}^a . In terms of these differentials we may define differential forms which are also $Sp(2)$ tensors, and *e.g.* a vector form ω^a is closed if $d^{\{a}\omega^{b\}} = 0$ and exact if $\omega^a = d^a\sigma$ where σ is an $Sp(2)$ scalar form of one lower degree. Now Y_{bcBC}^a may be expressed as coefficients of such a two-form (our conventions correspond to those of ref.[15]):

$$\omega_2^a = \frac{1}{2}Y_{bcBC}^a(-1)^{\varepsilon_C} d^c\Gamma^C \wedge d^b\Gamma^B - F_{bB}D^a d^b\Gamma^B. \quad (2.22)$$

where F_{aA} is an arbitrary vector field with Grassmann parity $\varepsilon(F_{aA}) = \varepsilon_A + 1$. D^a is a covariant exterior differential. This explains the existence of a Y_{bcBC}^a with the properties (2.2). It follows that the factors within the parenthesis of (2.20) are equal to the coefficients of $d^{\{f}\omega_2^{g\}}$. However, since we are unable to remove the tensors E^{FBb} , E^{GAa} and E^{CHc} eq.(2.20) does not imply that ω_2^a is closed. This is related to the fact that Y_{bcBC}^a is not unique. The last term in (2.22) is necessary in order to make ω_2^a exact when

$$Y_{bcBC}^a = \delta_b^a D_B F_{cC} - \delta_c^a D_C F_{bB}(-1)^{\varepsilon_B\varepsilon_C}, \quad (2.23)$$

where D_A is a covariant derivative with symmetric connections determined by

$$D_A E^{BCa} = 0, \quad [D_A, D_B] = 0 \quad (2.24)$$

The expression (2.23) is a particular solution of (2.20) for which we have

$$\omega_2^a = d^a\omega, \quad \omega = F_{aA}d^a\Gamma^A(-1)^{\varepsilon_A}. \quad (2.25)$$

In terms of V^a in (2.7), and by identifying the coefficients for different powers of ∂_A we find that conditions (2.16) imply

$$(\Delta^{\{a}\text{div}V^{b\}}) = 0, \quad (2.26)$$

$$\left(\Delta^{\{a}V^{B|b\}}(-1)^{\varepsilon_B}\right) + \frac{1}{2}\left(V^{\{a}(-1)^{\varepsilon_A}\left(\rho^{-1}\partial_A\rho E^{AB|b\}\right)\right) + \frac{1}{2}E^{BA\{a}(\partial_A\text{div}V^{b\}) = 0, \quad (2.27)$$

$$(V^{\{a}E^{AB|b\}}) = (-1)^{\varepsilon_A}V^{A\{a}\overset{\leftarrow}{\partial}_C E^{CB|b\}} - (A\leftrightarrow B)(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}, \quad (2.28)$$

and that (2.17) implies

$$[V^{\{a}, \text{div}V^{b\}}] = 0 \Leftrightarrow (V^{\{a}\text{div}V^{b\}}) = 0, \quad (2.29)$$

$$[V^a, V^b] = 0 \Leftrightarrow (V^{\{a}V^{Bb\}}) = 0. \quad (2.30)$$

Only the last conditions are conditions on V^a itself. Eq.(2.28) involves also E^{ABa} and (2.26),(2.27) and (2.29) are conditions on ρ , E^{ABa} and V^a .

In triplectic quantization there are two antibrackets, $(\ , \)^a$, $a = 1, 2$, and they are defined through the relation

$$\Delta^a(FG) = (\Delta^aF)G + F\Delta^aG(-1)^{\varepsilon(F)} + (F,G)^a(-1)^{\varepsilon(F)}, \quad (2.31)$$

where F and G are functions on \mathcal{M} . This combined with (2.6) leads to the following expression for the antibrackets in (2.31)

$$(F, G)^a = F \overleftarrow{\partial}_A E^{ABa} \partial_B G, \quad a = 1, 2. \quad (2.32)$$

From the properties (2.1), (2.18) and (2.19) of the tensor E^{ABa} it follows that these antibrackets satisfy the properties

$$(F, G)^a = -(G, F)^a (-1)^{(\varepsilon(F)+1)(\varepsilon(G)+1)}, \quad (2.33)$$

$$\varepsilon((F, G)^a) = \varepsilon(F) + \varepsilon(G) + 1, \quad (2.34)$$

$$(F, GH)^a = (F, G)^a H + G(F, H)^a (-1)^{(\varepsilon(F)+1)\varepsilon(G)}. \quad (2.35)$$

The identities $\Delta^{\{a} \Delta^{b\}}(FG) \equiv 0$ and $\Delta^{\{a} \Delta^{b\}}(FGH) \equiv 0$ together with (2.18), (2.19), and (2.32) yield furthermore

$$\Delta^{\{a}(F, G)^{b\}} = (\Delta^{\{a} F, G)^{b\}} + (F, \Delta^{\{a} G)^{b\}} (-1)^{\varepsilon(F)+1} \quad (2.36)$$

and

$$((F, G)^{\{a}, H)^{b\}} (-1)^{(\varepsilon(F)+1)(\varepsilon(H)+1)} + \text{cycle}(F, G, H) = 0 \quad (2.37)$$

respectively. Equation (2.37) is a version of the Jacobi identities satisfied by the two antibrackets. Notice also that the equality (2.28) is equivalent to the property

$$V^{\{a}(F, G)^{b\}} = (V^{\{a} F, G)^{b\}} + (F, V^{\{a} G)^{b\}} (-1)^{\varepsilon(F)+1}. \quad (2.38)$$

Thus, the V^a used here belongs to a more general class than the one of [1] where V^a was chosen to satisfy this relation without symmetrization in a and b . In fact, in [1] V^a were required to satisfy the relation

$$\epsilon_{ab}[\Delta_+^a, \Delta_-^b] = 0 \Leftrightarrow \epsilon_{ab}[\Delta^a, \hat{V}^b] = 0. \quad (2.39)$$

Although this was a convenient restriction it is not a necessary one. We shall refer to the V^a of [1] satisfying (2.39) as the symmetric V^a . (In [12] another argument for the general relation (2.38) was given.)

The above conditions do not determine a unique V^a . In fact, its natural arbitrariness is given by transformations of the form

$$V^a \mapsto V'^a = V^a + (\mathcal{H}, \cdot)^a \quad (2.40)$$

where $\mathcal{H}(\Gamma)$ is a bosonic function which is a solution of the equation

$$\frac{1}{2}(\mathcal{H}, \mathcal{H})^a + V^a \mathcal{H} = 0. \quad (2.41)$$

This restriction on \mathcal{H} follows from the conditions

$$[\hat{V}'^a, \hat{V}'^b] = 0, \quad (2.42)$$

while

$$[\Delta^{\{a}, \hat{V}'^b\}] = 0 \quad (2.43)$$

allows for arbitrary \mathcal{H} . The transformation (2.40) also induces

$$\frac{1}{2} \operatorname{div} V'^a = \frac{1}{2} \operatorname{div} V^a + \Delta^a \mathcal{H}. \quad (2.44)$$

The basic starting point in general triplectic quantization is the determination of the quantum master action $W(\Gamma; \hbar)$ where $W(\Gamma; \hbar)$ is expandable in powers of \hbar and is required to satisfy the quantum master equation

$$\Delta_+^a \exp \left\{ \frac{i}{\hbar} W \right\} = 0, \quad (2.45)$$

which may be equivalently written as

$$\frac{1}{2} (W, W)^a + V^a W = i\hbar \Delta^a W + \frac{1}{2} i\hbar \operatorname{div} V^a. \quad (2.46)$$

Notice that (2.14) are then just compatibility conditions for (2.45). The master action W should also be specified by imposing boundary conditions requiring them to coincide at $\hbar = 0$ with the original action S of the theory on some Lagrangian submanifold \mathcal{L}_0 :

$$W(\cdot; 0)|_{\mathcal{L}_0} = S(\cdot). \quad (2.47)$$

From (2.46) it follows that $\mathcal{H}(\Gamma)$ in the transformation (2.40) satisfies the classical master equation. This in turn implies that the quantum master equation (2.45) may also be written as

$$\Delta_+'^a \exp \left\{ \frac{i}{\hbar} W' \right\} = 0, \quad (2.48)$$

where

$$\Delta_+'^a = \Delta^a + \frac{i}{\hbar} \hat{V}'^a, \quad W' = W - \mathcal{H}. \quad (2.49)$$

Hence, the quantum master equation (2.45) is invariant under transformations of the form (2.40) provided the action W at the same time is shifted to W' given by (2.49). In particular, V^a may be transformed away completely if it is Hamiltonian with respect to the antibracket. In order to exclude this possibility we need to define the nondegeneracy properties of V^a .

In [1] an interesting Poisson bracket was introduced on a submanifold of \mathcal{M} in terms of which the nondegeneracy conditions on symmetric V^a operators was implicitly defined by the requirement that the Poisson bracket is nondegenerate on this submanifold. Such a Poisson bracket may also be introduced for more general V^a operators not satisfying (2.39). Consider the bracket

$$\{F, G\} \equiv \frac{1}{2} \epsilon_{ab} \left((F, V^b G)^a + (-1)^{\varepsilon(F)} (V^b F, G)^a \right), \quad (2.50)$$

which is a generalization of the corresponding definition in [1]. This bracket does not satisfy a Lie superalgebra on the whole triplectic manifold \mathcal{M} . We have to consider a submanifold of \mathcal{M} . We require therefore V^a to be such that there exists a class of functions \mathcal{F} satisfying

$$(F, G)^a = 0, \quad \forall F, G \in \mathcal{F}, \quad (2.51)$$

$$(F, V^{\{a} G^{b\}}) + (-1)^{\varepsilon(F)} (V^{\{a} F, G^{b\}}) = 0, \quad \forall F, G \in \mathcal{F}, \quad (2.52)$$

and also

$$\epsilon_{ab} \left((F, (\mathcal{H}, G)^b)^a + (-1)^{\varepsilon(F)} ((\mathcal{H}, F)^b, G)^a \right) = 0, \quad \forall F, G \in \mathcal{F}, \quad (2.53)$$

$$(F, (\mathcal{H}, G)^{\{a} b\}}) + (-1)^{\varepsilon(F)} ((\mathcal{H}, F)^{\{a} b\}}, G) = 0, \quad \forall F, G \in \mathcal{F}, \quad (2.54)$$

for bosonic functions \mathcal{H} satisfying the classical master equation (2.41). The definition (2.50) and the conditions (2.52) are then invariant under the transformations

$$V^a \mapsto V'^a = V^a + (\mathcal{H}, \cdot)^a \quad (2.55)$$

for those \mathcal{H} which satisfy (2.53), (2.54). If also

$$\{F, G\} \in \mathcal{F}, \quad \forall F, G \in \mathcal{F}, \quad (2.56)$$

and if we assume that there is a symmetric V^a in the class of V^a operators determine by the equivalence relation (2.55) then the bracket (2.50) is a graded Poisson bracket on \mathcal{F} , *i.e.* it satisfies the properties

$$\{G, F\} = -(-1)^{\varepsilon(F)\varepsilon(G)} \{F, G\}, \quad (2.57)$$

$$\{F, GH\} = \{F, G\}H + G\{F, H\}(-1)^{\varepsilon(F)\varepsilon(G)}, \quad (2.58)$$

$$\{F, \{G, H\}\}(-1)^{\varepsilon(F)\varepsilon(H)} + \text{cycle}(F, G, H) = 0. \quad (2.59)$$

Notice that (2.57) is satisfied by (2.50) by construction due to (2.33), and that (2.58) follows from (2.51). The proof of (2.59) is given in [1]. The nondegeneracy of V^a should furthermore be such that the functions in \mathcal{F} span a submanifold \mathcal{L}_1 of \mathcal{M} with dimension $2N$ ($\dim \mathcal{M} = 6N$) and that the Poisson bracket (2.50) is nondegenerate on \mathcal{L}_1 . (In fact, the existence of a symmetric V^a *requires* these nondegeneracy properties.) This generalizes the conditions in [1].

3 New version of the Δ_{\pm}^a - operators and the origin of V^a

The generating operators Δ_{\pm}^a introduced in (2.5) are composed of two different objects, Δ^a and V^a , which have no natural relation between them apart from the nilpotency conditions (2.14) of Δ_{\pm}^a . In this section we propose a unified expression for Δ_{\pm}^a which is more invariant in the sense that (2.14) allows for transformations of the form (2.40) for arbitrary bosonic functions \mathcal{H} . In addition it allows us to demonstrate the existence of

V^a by deriving a geometrical explanation for its origin. The resulting more explicit form of V^a is also a covariant object which incorporates the original expression in Darboux coordinates obtained from the Hamiltonian treatment in [2]-[5]. Our unified expression is

$$\Delta_{\pm}^a = \frac{1}{2}(-1)^{\varepsilon_A} \left(\rho^{-1} \partial_A \circ \rho \pm \frac{i}{\hbar} F_A \right) E^{ABa} \left(\partial_B \pm \frac{i}{\hbar} F_B \right) \quad (3.1)$$

where F_A ($\varepsilon(F_A) = \varepsilon_A$) is an additional connection to a trivial volume connection. F_A transforms as a vector field. (One may compare this expression with the general ansatz for Δ in the antisymplectic case given in [16].) The operators (3.1) are also hermitian with respect to the scalar product (2.8) provided E^{ABa} transforms according to (2.13) and F_A according to

$$F_A = I_A^D F_D^* (-1)^{\varepsilon_D}, \quad F_A^* = F_D I_A^{*D} (-1)^{\varepsilon_D} \quad (3.2)$$

under the complex-conjugate condition (2.9). Expanding (3.1) in i/\hbar we find

$$\Delta_{\pm}^a = \Delta^a \pm \frac{i}{\hbar} (V^a + \frac{1}{2} \text{div} V^a) + \left(\frac{i}{\hbar} \right)^2 \frac{1}{2} (-1)^{\varepsilon_A} F_A E^{ABa} F_B \quad (3.3)$$

where

$$V^a = V^{Aa} (-1)^{\varepsilon_A} \partial_A, \quad V^{Aa} \equiv E^{ABa} F_B = F_B E^{BAa} (-1)^{\varepsilon_A + \varepsilon_B}, \quad (3.4)$$

which, thus, is a special realization of the general V^a of the previous section. As mentioned above this special covariant V^a also incorporates the original expressions in special coordinates. However, due to the presence of the $(i/\hbar)^2$ terms in (3.3) this Δ_{\pm}^a does not coincide with the Δ_{\pm}^a of the previous section. Ignoring this fact for the moment we consider the conditions

$$[\Delta_{\pm}^a, \Delta_{\pm}^b] = 0 \quad \Leftrightarrow \quad \Delta_{\pm}^{\{a} \Delta_{\pm}^{b\}} = 0. \quad (3.5)$$

To the zeroth order in i/\hbar these conditions are identical to (2.15) or equivalently (2.18) and (2.19), and to the first order they yield (2.16) or equivalently (2.26)-(2.28). Inserting (3.4) into (2.28) and making use of (2.18) we find that (2.28) reduces to

$$E^{AC\{a} (\partial_C F_D - \partial_D F_C (-1)^{\varepsilon_C \varepsilon_D}) (-1)^{\varepsilon_D} E^{DB|b\}} = 0. \quad (3.6)$$

Conditions (2.26)-(2.27) are however complex relations between F_A , E^{ABa} and ρ which are difficult to solve in general coordinates. To the second and third order in (i/\hbar) eq.(3.5) yields

$$\frac{1}{2} [\Delta^{\{a}, (-1)^{\varepsilon_A} F_A E^{AB|b\}} F_B] + [V^{\{a}, V^{b\}}] + \frac{1}{2} [V^{\{a}, \text{div} V^{b\}}] = 0, \quad (3.7)$$

$$[V^{\{a}, (-1)^{\varepsilon_A} F_A E^{AB|b\}} F_B] = (-1)^{\varepsilon_A} (V^{\{a} F_A E^{AB|b\}} F_B) = 0. \quad (3.8)$$

Identifying different powers of ∂_A in (3.7) it splits into the following conditions

$$(\Delta^{\{a} (-1)^{\varepsilon_A} F_A E^{AB|b\}} F_B) + (V^{\{a} \text{div} V^{b\}}) = 0, \quad (3.9)$$

$$\frac{1}{2} E^{AB\{a} \partial_B (-1)^{\varepsilon_C} F_C E^{CD|b\}} F_D + V^{\{a} V^{A|b\}} (-1)^{\varepsilon_A} = 0. \quad (3.10)$$

Remarkably enough conditions (3.8)-(3.10) turn out to be identically satisfied. Eqs.(3.8) and (3.10) are satisfied when (3.6) and (2.18) are imposed, and (3.9) follows from (2.27). Thus, eq.(3.6) are the only necessary conditions on F_A in order to satisfy (3.5) apart from the involved relations (2.26)-(2.27) which might impose further restrictions. This high degree of symmetry is also reflected in the property that all conditions coming from (3.5) are invariant under the transformations

$$F_A \rightarrow F_A + \partial_A \mathcal{H} \quad (3.11)$$

for *any* bosonic function $\mathcal{H}(\Gamma)$. Consider now the quantum master equation

$$\Delta_+^a \exp\left(\frac{i}{\hbar} W\right) = 0, \quad (3.12)$$

which may be written as

$$\begin{aligned} & \frac{1}{2}(W, W)^a + (-1)^{\varepsilon_A} (E^{ABa} F_B) \partial_A W + \frac{1}{2} (-1)^{\varepsilon_A} F_A E^{ABa} F_B = \\ & = i\hbar \Delta^a W + \frac{1}{2} i\hbar (-1)^{\varepsilon_A} \rho^{-1} \partial_A \rho E^{ABa} F_B. \end{aligned} \quad (3.13)$$

These equations are invariant under (3.11) provided we also shift the master action W by

$$W \mapsto W - \mathcal{H}. \quad (3.14)$$

For (3.12) this is obvious from the form (3.1) of Δ_+^a . The classical part of the master equation, *i.e.* the left-hand side of (3.13), does not agree with the expression obtained from the Hamiltonian treatment. This forces us to impose the auxiliary conditions

$$(-1)^{\varepsilon_A} F_A E^{ABa} F_B = 0. \quad (3.15)$$

They remove the last terms in (3.3) and make our unified expression consistent with the general conditions on Δ_{\pm}^a given in the previous section. This isotropy condition on F_A is invariant under the transformation (3.11) provided the bosonic function $\mathcal{H}(\Gamma)$ satisfies the classical master equation (2.41) with V^a given by (3.4). Thus, the invariance transformations (3.11) have now been reduced to the arbitrariness (2.40) of V^a in the previous section.

One may now notice that the vorticity entering the parenthesis in (3.6) can be considered as coefficients of $d^{\{a} \omega_1^{b\}}$ where ω_1^a is the one-form

$$\omega_1^a \equiv F_A d^a \Gamma^A. \quad (3.16)$$

We have

$$d^{\{a} \omega_1^{b\}} = -\frac{1}{2} (\partial_A F_B - \partial_B F_A (-1)^{\varepsilon_A \varepsilon_B}) (-1)^{\varepsilon_A} d^{\{b} \Gamma^B \wedge d^a \Gamma^A. \quad (3.17)$$

Notice, however, that ω_1^a is not closed since we cannot remove $E^{CAa/b}$ and $E^{BDb/a}$ in (3.6) (cf. (2.20)). The deviation from closeness is measured by the tensor

$$\omega^{AB} \equiv \frac{1}{2} \epsilon_{ab} E^{ACa} (\partial_C F_D - \partial_D F_C (-1)^{\varepsilon_C \varepsilon_D}) (-1)^{\varepsilon_D} E^{DBb}. \quad (3.18)$$

We know from our expressions of V^a in terms of Darboux coordinates that ω^{AB} is different from zero. A condition which is invariant under (3.11) and which comply with these results are

$$\text{rank } \omega^{AB} = 2N, \quad (\dim \mathcal{M} = 6N). \quad (3.19)$$

This is just a nondegeneracy condition on F_A . It means that F_A must be nontrivial ($F_A \neq \partial_A \mathcal{H}$) and that V^a cannot be transformed away in the quantum master equation (2.46). It is related to our condition of a nondegenerate Poisson bracket given in [1] and in the previous section. Indeed if we specify the submanifold \mathcal{L}_1 of \mathcal{M} appropriate for the Poisson bracket (2.50) by $F_A = 0$ we have

$$\{\Gamma^A, \Gamma^B\} \Big|_{\mathcal{L}_1} = \omega^{AB}. \quad (3.20)$$

Another invariant condition which comply with the results in Darboux coordinates is

$$\text{rank } F_{AB} = 4N, \quad F_{AB} \equiv \partial_A F_B - \partial_B F_A (-1)^{\varepsilon_A \varepsilon_B}. \quad (3.21)$$

A still stronger invariant condition also true in special coordinates is

$$(-1)^{\varepsilon_B} F_{AB} E^{BCa} F_{CD} = 0, \quad (3.22)$$

which has the structure of the isotropy condition (3.15). It follows immediately from (3.22) that

$$\omega^{AB} F_{BC} = 0. \quad (3.23)$$

We could require for ω^{AB} and F_{AB} to form complete basis for each others nullvectors so that

$$\text{rank } \omega^{AB} + \text{rank } F_{AB} = 6N, \quad (3.24)$$

which is consistent with (3.19) and (3.21).

If the condition (3.22) is accepted we may introduce a new Poisson bracket defined on the whole triplectic manifold \mathcal{M} by

$$\{F, G\} \equiv F \overleftarrow{\partial}_A \omega^{AB} \partial_B G \quad (3.25)$$

for *arbitrary* functions $F(\Gamma)$ and $G(\Gamma)$. It satisfies the antisymmetry (2.57) and the Leibniz rule (2.58). The Jacobi identities (2.59) follow from the properties

$$\omega^{AD} \partial_D \omega^{BC} (-1)^{\varepsilon_A \varepsilon_C} + \text{cycle}(A, B, C) = 0, \quad (3.26)$$

which are straight-forward to prove. First one has to use the cyclic identities

$$\partial_A F_{BC} (-1)^{\varepsilon_A \varepsilon_C} + \text{cycle}(A, B, C) \equiv 0, \quad (3.27)$$

then conditions (3.6) and (3.22) to remove the derivatives on F_{AB} . Finally one has to use (2.18) to reproduce (3.26). The Poisson bracket (3.25) is degenerate on \mathcal{M} . It is nondegenerate only on the submanifold \mathcal{L}_1 defined by $F_A = 0$ which is of dimension $2N$. Notice that (3.25) is equivalent to (2.50) only on \mathcal{L}_1 . Notice also that the condition (2.52) reduces to (3.6) on \mathcal{L}_1 . In distinction to the Poisson bracket (2.50) the new bracket (3.25)

is invariant under the gradient shifts (3.11) for *any* bosonic generator \mathcal{H} . The properties of the submanifold \mathcal{L}_1 are in general rather complicated as it may be seen from the generalized involution relations for F_A which follow from (3.15) and (3.22), *i.e.*

$$(F_A, F_B)^a + (-1)^{\varepsilon_C} F_{AC}(\Gamma^C, F_B)^a - (-1)^{\varepsilon_C} F_{BC}(\Gamma^C, F_A)^a (-1)^{(\varepsilon_A+1)(\varepsilon_B+1)} = \mathcal{U}_{AB}^{C_a} F_C. \quad (3.28)$$

Notice that these relations are only invariant under the same restricted shifts (3.11) under which the isotropy conditions (3.15) are invariant.

The basic reason why we have V^a operators within triplectic quantization and not corresponding V operators within the antisymplectic quantization is the fact that (3.6) allows for nontrivial solutions. In antisymplectic quantization the corresponding condition to (3.6) may be transformed to $\partial_C F_D - \partial_D F_C (-1)^{\varepsilon_C \varepsilon_D} = 0$ with only trivial solutions which may be transformed away in the master equation.

4 Gauge fixing in general triplectic quantization

The path integral in general triplectic Lagrangian quantization is proposed to be [1]

$$Z = \int \exp \left\{ \frac{i}{\hbar} [W + X] \right\} \rho(\Gamma) [d\Gamma] [d\lambda], \quad (4.1)$$

where $W(\Gamma; \hbar)$ is the quantum master action which satisfies (2.46) and where $X(\Gamma, \lambda; \hbar)$ is a gauge fixing action which depends on the parametric variables λ^α , $\alpha = 1, \dots, N$, which are generalized Lagrange multipliers for hypergauge conditions. In [1] it was shown that provided X satisfies the "weak" quantum master equation

$$\left(\Delta^a - \frac{i}{\hbar} \hat{V}^a - \frac{i}{\hbar} (-1)^{\varepsilon_\alpha} R^{\alpha a} \partial_\alpha + (-1)^{\varepsilon_\alpha} \partial_\alpha R^{\alpha a} \right) \exp \left(\frac{i}{\hbar} X \right) = 0, \quad (4.2)$$

or equivalently,

$$\frac{1}{2} (X, X)^a - i\hbar \Delta^a X - V^a X + \frac{1}{2} i\hbar \operatorname{div} V^a - X \overleftarrow{\partial}_\alpha R^{\alpha a} + i\hbar R^{\alpha a} \overleftarrow{\partial}_\alpha = 0, \quad (4.3)$$

where $\partial_\alpha \equiv \partial/\partial\lambda^\alpha$ and $\varepsilon_\alpha \equiv \varepsilon(\lambda^\alpha)$, then (4.1) is invariant under the general canonical transformation

$$\delta\Gamma^A = (\Gamma^A, -W + X)^a \mu_a - 2V^{Aa} \mu_a, \quad \delta\lambda^\alpha = -2R^{\alpha a} \mu_a, \quad (4.4)$$

where μ_a are two fermionic constants. The consistency conditions for (4.2) are

$$\frac{\partial}{\partial\lambda^\alpha} \left\{ \left(i\hbar (\Delta^{\{a} R^{\alpha b\}}) + V^{\{a} R^{\alpha b\}} - (X, R^{\alpha\{a} \}^b) + (-1)^{\varepsilon_\beta} R^{\beta\{a} \} \partial_\beta R^{\alpha b\}} \right) e^{\frac{i}{\hbar} X} \right\} = 0, \quad (4.5)$$

which are obtained by applying the operator

$$\Delta^b - \frac{i}{\hbar} V^b + \frac{i}{\hbar} (X, \cdot)^b - \frac{i}{\hbar} (-1)^{\varepsilon_\alpha} R^{\alpha b} \partial_\alpha \quad (4.6)$$

to the left-hand side of eq.(4.2) and symmetrizing in a and b . As shown in [1] (4.5) may be solved within an extended formalism which is obtained as follows: Introduce a linear

space Λ spanned by $(\lambda^\alpha, \lambda_{\alpha a}^*, \bar{\lambda}_\alpha, \eta^{\alpha a})$, with $\epsilon(\lambda^\alpha) = \epsilon(\bar{\lambda}_\alpha) = \epsilon_\alpha$, $\epsilon(\lambda_{\alpha a}^*) = \epsilon(\eta^{\alpha a}) = \epsilon_\alpha + 1$, and define an extended triplectic manifold $\widetilde{\mathcal{M}} = \mathcal{M} \times \Lambda$. On $\widetilde{\mathcal{M}}$, one may then introduce the operators

$$\Delta_{\text{ext}}^a = \Delta^a + (-1)^{\epsilon_\alpha} \frac{\partial}{\partial \lambda^\alpha} \frac{\partial}{\partial \lambda_{\alpha a}^*} + (-1)^{\epsilon_\alpha + 1} \epsilon^{ab} \frac{\partial}{\partial \bar{\lambda}_\alpha} \frac{\partial}{\partial \eta^{\alpha b}} \quad (4.7)$$

and the corresponding antibrackets

$$(F, G)_{\text{ext}}^a = (F, G)^a + \left(F \frac{\overleftarrow{\partial}}{d\lambda^\alpha} \frac{\partial}{\partial \lambda_{\alpha a}^*} G + \epsilon^{ab} F \frac{\overleftarrow{\partial}}{\partial \eta^{\alpha b}} \frac{\partial}{\partial \bar{\lambda}_\alpha} G - (-1)^{(\epsilon(F)+1)(\epsilon(G)+1)} (F \leftrightarrow G) \right). \quad (4.8)$$

The vector fields V^a may then be extended to $\widetilde{\mathcal{M}}$ *e.g.* as follows (which generalizes [1])

$$\mathcal{V}^a \equiv V^a - (1 - \beta) \epsilon^{ab} \lambda_{\alpha b}^* \frac{\partial}{\partial \bar{\lambda}_\alpha} + \beta (-1)^{\epsilon_\alpha} \eta^{\alpha a} \frac{\partial}{\partial \lambda^\alpha}, \quad (4.9)$$

which satisfy the necessary conditions $[\hat{\mathcal{V}}^{\{a}, \Delta_{\text{ext}}^{b\}}] = 0$ etc. in accordance with the conditions given in the previous section for any real constant β .

Now we can introduce an extended quantum master equation

$$\left(\Delta_{\text{ext}}^a - \frac{i}{\hbar} \hat{\mathcal{V}}^a \right) \exp \left(\frac{i}{\hbar} \mathcal{X} \right) = 0, \quad \hat{\mathcal{V}}^a \equiv \mathcal{V}^a + \frac{1}{2} \text{div } \mathcal{V}^a, \quad (4.10)$$

or equivalently,

$$\frac{1}{2} (\mathcal{X}, \mathcal{X})_{\text{ext}}^a - \mathcal{V}^a \mathcal{X} - i\hbar \Delta_{\text{ext}}^a \mathcal{X} + \frac{1}{2} i\hbar \text{div } \mathcal{V}^a = 0. \quad (4.11)$$

The "weak" master equation (4.2) follows then from (4.10) or (4.11) once we take \mathcal{X} to be

$$\mathcal{X}(\Gamma, \lambda, \lambda^*, \bar{\lambda}, \eta) = \widetilde{X}(\Gamma, \lambda, \lambda^*, \bar{\lambda}) + \beta \lambda_{\alpha a}^* \eta^{\alpha a} \quad (4.12)$$

and expand \widetilde{X} as follows

$$\begin{aligned} \widetilde{X}(\Gamma, \lambda, \lambda^*, \bar{\lambda}) &= X(\Gamma, \lambda) - \lambda_{\alpha a}^* R^{\alpha a}(\Gamma, \lambda) - \bar{\lambda}_\alpha \bar{R}^\alpha(\Gamma, \lambda) + \\ &+ \frac{1}{2} \lambda_{\alpha a}^* \lambda_{\beta b}^* F^{\alpha a; \beta b} + \frac{1}{2} \bar{\lambda}_\alpha \bar{\lambda}_\beta \bar{F}^{\alpha \beta} + \bar{\lambda}_\beta \lambda_{\alpha a}^* E^{\alpha a; \beta} + \text{higher orders in } \lambda^*, \bar{\lambda}. \end{aligned} \quad (4.13)$$

Eq.(4.10) implies then *e.g.* to the first order in λ^* :

$$\begin{aligned} i\hbar \Delta^a R^{\alpha b} + V^a R^{\alpha b} + (-1)^{\epsilon_\beta} R^{\beta a} \partial_\beta R^{\alpha b} - (X, R^{\alpha b})^a + (-1)^{\epsilon_\alpha} \epsilon^{ab} \bar{R}^\alpha &= \\ = (-1)^{\epsilon_\beta} (i\hbar \partial_\beta F^{\alpha b; \beta a} - \partial_\beta X F^{\alpha b; \beta a}), \end{aligned} \quad (4.14)$$

which solves the consistency conditions (4.5) upon symmetrization.

The integral (4.1) can now be reformulated on the extended triplectic manifold using the extended master action \mathcal{X} as follows [1]

$$\begin{aligned} Z &= \int \exp \left\{ \frac{i}{\hbar} [W + \widetilde{X} + \lambda_{\alpha a}^* \eta^{\alpha a} + \bar{\lambda}_\alpha \xi^\alpha] \right\} \rho(\Gamma) \underbrace{[d\Gamma][d\lambda][d\lambda^*][d\bar{\lambda}][d\eta][d\xi]}_{[d\widetilde{\Gamma}]} \\ &= \int \exp \left\{ \frac{i}{\hbar} [\mathcal{W} + \mathcal{X}] \right\} \rho(\widetilde{\Gamma}) [d\widetilde{\Gamma}] [d\xi] \end{aligned} \quad (4.15)$$

where $\xi^\alpha \equiv \lambda^{(1)\alpha}$ are Lagrange multipliers of the *next-level* theory, and where

$$\mathcal{W} = W + (1 - \beta)\lambda_{\alpha a}^* \eta^{\alpha a} + \bar{\lambda}_\alpha \xi^\alpha \quad (4.16)$$

is a solution on $\widetilde{\mathcal{M}}$ to the strong master equation:

$$\left(\Delta_{\text{ext}}^a + \frac{i}{\hbar} \hat{\mathcal{V}}^a \right) \exp \left(\frac{i}{\hbar} \mathcal{W} \right) = 0, \quad (4.17)$$

or equivalently,

$$\frac{1}{2}(\mathcal{W}, \mathcal{W})_{\text{ext}}^a + \mathcal{V}^a \mathcal{W} - i\hbar \Delta_{\text{ext}}^a \mathcal{W} - \frac{1}{2}i\hbar \text{div } \mathcal{V}^a = 0. \quad (4.18)$$

Notice that at this level \mathcal{W} has become a gauge fixing action (depends on ξ^α) and \mathcal{X} a master action.

The path integral (4.15) is invariant under the canonical transformation

$$\begin{aligned} \widetilde{\Gamma}^I &= (\widetilde{\Gamma}^I, -\mathcal{W} + \mathcal{X})_{\text{ext}}^a \mu_a - 2\mathcal{V}^I a \mu_a, \quad \delta\lambda^{(1)\alpha} \equiv \delta\xi^\alpha = 0 \\ I &= 1, \dots, 12N, \quad \alpha = 1, \dots, N \end{aligned} \quad (4.19)$$

where μ^a again are two fermionic constants. One may notice that the natural arbitrariness in \mathcal{V}^a as described by the transformations

$$\mathcal{V}'^a = \mathcal{V}^a + (\mathcal{H}, \cdot)_{\text{ext}}^a, \quad \frac{1}{2}(\mathcal{H}, \mathcal{H})_{\text{ext}}^a + \mathcal{V}^a \mathcal{H} = 0, \quad (4.20)$$

which generalize formulas (2.40) and (2.41), and where $\mathcal{H}(\widetilde{\Gamma})$ here is a bosonic function on $\widetilde{\mathcal{M}}$, imply that the master equations (4.10) and (4.17) are invariant under $\mathcal{V}^a \mapsto \mathcal{V}'^a$ provided \mathcal{W} and \mathcal{X} at the same time are shifted according to $\mathcal{W} \mapsto \mathcal{W}' = \mathcal{W} - \mathcal{H}$, $\mathcal{X} \mapsto \mathcal{X}' = \mathcal{X} + \mathcal{H}$. In (4.15) we get then $\mathcal{W} + \mathcal{X} \mapsto \mathcal{W}' + \mathcal{X}' = \mathcal{W} + \mathcal{X}$ and the invariance transformations (4.19) become exactly the same expressions with \mathcal{W} , \mathcal{X} and \mathcal{V}^a replaced by \mathcal{W}' , \mathcal{X}' and \mathcal{V}'^a . In fact, the arbitrariness in \mathcal{V}^a given in (4.9) as represented by the arbitrary constant β is of the general form (4.20) since

$$\mathcal{V}^a \equiv V^a - \epsilon^{ab} \lambda_{\alpha b}^* \frac{\partial}{\partial \bar{\lambda}_\alpha} + (\beta \lambda_{\alpha b}^* \pi^{\alpha b}, \cdot)^a. \quad (4.21)$$

The solutions of (4.10) and (4.17) are mapped on new solutions, $\mathcal{X} \mapsto \mathcal{X}'$ and $\mathcal{W} \mapsto \mathcal{W}'$, according to the formulas [4]

$$\exp \left(\frac{i}{\hbar} \mathcal{X}' \right) = \exp \left(\epsilon_{ab} \frac{\hbar}{i} [\Delta_{\text{ext}}^a - \frac{i}{\hbar} \hat{\mathcal{V}}^a, [\Delta_{\text{ext}}^b - \frac{i}{\hbar} \hat{\mathcal{V}}^b, \Phi]] \right) \exp \left(\frac{i}{\hbar} \mathcal{X} \right), \quad (4.22)$$

$$\exp \left(\frac{i}{\hbar} \mathcal{W}' \right) = \exp \left(\epsilon_{ab} \frac{\hbar}{i} [\Delta_{\text{ext}}^a + \frac{i}{\hbar} \hat{\mathcal{V}}^a, [\Delta_{\text{ext}}^b + \frac{i}{\hbar} \hat{\mathcal{V}}^b, \Psi]] \right) \exp \left(\frac{i}{\hbar} \mathcal{W} \right), \quad (4.23)$$

where Φ and Ψ are arbitrary *bosonic* functions or operators. In [1] a proof was given that the path integral (4.15) is independent of the natural arbitrariness of the solutions of (4.10) given in (4.22). This proof was based on the transformation (4.19). The path integral is also independent of the arbitrariness (4.23) respecting the boundary conditions (2.47).

5 Second class hyperconstraints

As shown in [1] if we set the classical limit of the gauge fixing action X in (4.1) to be linear in λ^α , *i.e.*

$$X(\Gamma, \lambda; 0) = G_\alpha(\Gamma)\lambda^\alpha + Z(\Gamma), \quad (5.1)$$

then the "weak" master equation (4.2) requires

$$(G_\alpha, G_\beta)^a = G_\gamma U_{\alpha\beta}^{\gamma a}, \quad (5.2)$$

$$(Z, G_\alpha)^a - V^a G_\alpha = G_\beta U_\alpha^{\beta a}, \quad (5.3)$$

$$\frac{1}{2}(Z, Z)^a - V^a Z = \frac{1}{2}G_\gamma U^{\gamma a}, \quad (5.4)$$

where $U_{\alpha\beta}^{\gamma a}$, $U_\alpha^{\beta a}$ and $U^{\gamma a}$ are coefficients for different powers of λ^α in $R^{\gamma a}(\Gamma, \lambda; 0)$ (see [1]). One may notice that Z may be transformed away in X by a transformation of the form (4.20) after which (5.4) is eliminated. G_α are here hypergauge generators which may be viewed as first class hyperconstraints due to (5.2). By means of the nondegeneracy concept introduced for E^{ABa} in (2.2)-(2.4) we may also introduce second class hyperconstraints within the triplectic framework. We have then the following generalization of the corresponding antisymplectic treatment of [16]: Call $\Theta^\alpha(\Gamma)$, $\alpha = 1, \dots, 6K$, second class constraints if there exists a $Y_{ab\beta\gamma}^c$ satisfying

$$E^{\alpha\beta a} Y_{ab\beta\gamma}^c = \delta_\alpha^\gamma \delta_b^c, \quad E^{\alpha\beta a} \equiv (\Theta^\alpha, \Theta^\beta)^a, \quad (5.5)$$

and

$$\varepsilon(Y_{ab\beta\gamma}^c) = \varepsilon(\Theta^\beta) + \varepsilon(\Theta^\gamma) + 1, \quad Y_{bc\beta\gamma}^a = -Y_{cb\gamma\beta}^a (-1)^{\varepsilon(\Theta^\beta)\varepsilon(\Theta^\gamma)}. \quad (5.6)$$

We may then define the triplectic counterpart of the Dirac bracket by

$$(A, B)_{(D)}^a \equiv (A, B)^a - (A, \Theta^\beta)^b Y_{bc\beta\gamma}^a (\Theta^\gamma, B)^c. \quad (5.7)$$

These generalized Dirac brackets satisfy all the required antibracket properties (2.32)-(2.37) and

$$(A, \Theta^\beta)_{(D)}^a \equiv 0. \quad (5.8)$$

Notice now that

$$E_{(D)}^{ABa} \equiv (\Gamma^A, \Gamma^B)_{(D)}^a \quad (5.9)$$

is an example of a degenerate tensor. Still we may introduce nilpotent differential operators

$$\begin{aligned} \Delta_{\pm(D)}^a &= \Delta_{(D)}^a \pm \frac{i}{\hbar}(V_{(D)}^a + \frac{1}{2}\text{div}V_{(D)}^a), \\ \Delta_{(D)}^a &= \frac{1}{2}(-1)^{\varepsilon_A} \rho_{(D)}^{-1} \partial_A \circ \rho_{(D)} E_{(D)}^{ABa} \partial_B, \\ V_{(D)}^a &= (-1)^{\varepsilon_A} V_{(D)}^{Aa} \partial_A, \quad \text{div}V_{(D)}^a \equiv \rho_{(D)}^{-1} \left(\partial_A \rho_{(D)} V_{(D)}^{Aa} \right) (-1)^{\varepsilon_A}, \end{aligned} \quad (5.10)$$

which satisfy the required nilpotency conditions corresponding to (2.14), and which, furthermore, satisfy

$$\Delta_{(D)}^a \Theta^\alpha = 0, \quad V_{(D)}^a \Theta^\alpha = 0 \quad \Rightarrow \quad \Delta_{\pm(D)}^a \Theta^\alpha = 0. \quad (5.11)$$

The last properties follow from (5.8) if we make use of the natural representation

$$V_{(D)}^a = (-1)^{\varepsilon_B} F_B E_{(D)}^{BAa} \partial_A, \quad (5.12)$$

in accordance with (3.4). In this latter case $\Delta_{\pm(D)}^a$ may also be written in the unified form (3.1) with E^{ABa} replaced by $E_{(D)}^{ABa}$.

The second class constraints, $\Theta^\alpha = 0$, should eliminate $6K$ degrees of freedom from the original triplectic manifold \mathcal{M} ($\dim \mathcal{M} = 6N$) leaving a nondegenerate triplectic manifold $\mathcal{M}_{(D)}$ of dimension $6(N - K)$. Corresponding to (4.1) we have then the gauge fixed path integral expression

$$Z = \int \exp \left\{ \frac{i}{\hbar} [W_{(D)} + X_{(D)}] \right\} d\mu_{(D)}(\Gamma) [d\lambda], \quad (5.13)$$

where $W_{(D)}$ and $X_{(D)}$ satisfy the master eqs.(2.46) and (4.3) with all operators replaced by their Dirac counterparts, and where the volume measure is given by

$$d\mu_{(D)}(\Gamma) \equiv \rho_{(D)}(\Gamma) \prod_{\alpha} \delta(\Theta^\alpha) [d\Gamma]. \quad (5.14)$$

The path integral (5.13) is then invariant under the general canonical transformation

$$\delta\Gamma^A = (\Gamma^A, -W_{(D)} + X_{(D)})_{(D)}^a \mu_a - 2V_{(D)}^{Aa} \mu_a, \quad \delta\lambda^\alpha = -2R^{\alpha a} \mu_a, \quad (5.15)$$

where μ_a are two fermionic constants.

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Appendix

Some basic objects as represented in terms of the Darboux coordinates

Let us consider the special Darboux coordinates of refs.[5] and [1], *i.e.*

$$\Gamma^A = (\phi^\alpha, \phi_{\alpha a}^*), \quad \alpha = 1, \dots, 2N, \quad \varepsilon(\phi^\alpha) = \varepsilon_\alpha, \quad \varepsilon(\phi_{\alpha a}^*) = \varepsilon_\alpha + 1, \quad (\text{A.1})$$

where ϕ^α are field variables and $\phi_{\alpha a}^*$ antifields. In terms of these coordinates we have

$$\Delta^a = (-1)^{\varepsilon_\alpha} \frac{\partial}{\partial \phi^\alpha} \frac{\partial}{\partial \phi_{\alpha a}^*}, \quad (\text{A.2})$$

$$(F, G)^a = F \frac{\overleftarrow{\partial}}{\partial \phi^\alpha} \frac{\partial}{\partial \phi_{\alpha a}^*} G - F \frac{\overleftarrow{\partial}}{\partial \phi_{\alpha a}^*} \frac{\partial}{\partial \phi^\alpha} G, \quad (\text{A.3})$$

and

$$V^a = \epsilon^{ab} \phi_{\alpha b}^* \kappa^{\alpha\beta} \frac{\partial}{\partial \phi^\beta}, \quad (\text{A.4})$$

where $\kappa^{\alpha\beta}$ is an off-diagonal constant matrix. The last expression includes the V^a operators given in refs.[2]-[5].

The basic antitriplectic metric E^{ABa} is given by

$$E^{ABc} = \begin{pmatrix} E^{\alpha\beta c} & E^\alpha_{\beta b}{}^c \\ E_{\alpha a}{}^{\beta c} & E_{\alpha a \beta b}{}^c \end{pmatrix} = \begin{pmatrix} 0 & \delta_\beta^\alpha \delta_b^c \\ -\delta_\alpha^\beta \delta_a^c & 0 \end{pmatrix}, \quad (\text{A.5})$$

and the inverses Y_{dfAB}^c are in turn

$$Y_{dfAB}^c = \begin{pmatrix} Y_{df\alpha\beta}^c & Y_{df\alpha}^c{}^{\beta b} \\ Y_{df}^c{}_{\alpha a}{}_\beta & Y_{df}^c{}_{\alpha a \beta b} \end{pmatrix} = \begin{pmatrix} 0 & -\delta_\alpha^\beta \delta_d^b \delta_f^c \\ \delta_\beta^\alpha \delta_f^a \delta_d^c & Y_{df}^c{}_{\alpha a \beta b} \end{pmatrix}, \quad (\text{A.6})$$

where $Y_{df}^c{}_{\alpha a \beta b}$ is not uniquely determined. The conditions (2.2) and (2.3) only require

$$Y_{af}^c{}_{\alpha a \beta b} = 0. \quad (\text{A.7})$$

The two-form ω_2^a defined by (2.22) is then ($D^a = d^a$ in Darboux coordinates)

$$\begin{aligned}\omega_2^a = & (-1)^{\varepsilon_\beta} d^a \phi_{\beta b}^* \wedge d^b \phi^\beta - \frac{1}{2} Y_{df}^{a\beta b\gamma c} (-1)^{\varepsilon_\gamma} d^f \phi_{\gamma c}^* \wedge d^d \phi_{\beta b}^* - \\ & - F_b^{\gamma c} d^a d^b \phi_{\gamma c}^* - F_{b\beta} d^a d^b \phi^\beta,\end{aligned}\quad (\text{A.8})$$

where $F_b^{\gamma c}$ and $F_{b\beta}$ are also undetermined. This arbitrariness may be fixed if we assume that ω_2^a is exact for Darboux coordinates. From (2.23), (2.25) we have then more precisely

$$\omega_2^a = d^a \omega = (-1)^{\varepsilon_\beta} d^a \phi_{\beta b}^* \wedge d^b \phi^\beta - \phi_{\beta b}^* d^a d^b \phi^\beta, \quad \omega = \phi_{\alpha a}^* d^a \phi^\alpha (-1)^{\varepsilon_\alpha}. \quad (\text{A.9})$$

If we consider the unified expression (3.1) of section 3 we may easily derive the general nontrivial form of V^a in terms of the Darboux coordinates. We start then from the natural representation (3.4) of V^a , *i.e.*

$$V^a = (-1)^{\varepsilon_B} F_B E^{BAa} \partial_A = (-1)^{\varepsilon_\alpha} F_\alpha \frac{\partial}{\partial \phi_{\alpha a}^*} + (-1)^{\varepsilon_\alpha} F^{\alpha a} \frac{\partial}{\partial \phi^\alpha}, \quad (\text{A.10})$$

where we in the last equality have made use of (A.1) and (A.5). Inserting F_A into the condition (3.6) using (A.5) we find that the nontrivial part of F_A ($\neq \partial_A \mathcal{H}$) is contained in the antifield components and that these components only depend on the antifields, *i.e.* $F^{\alpha a}(\phi_{\beta b}^*)$. A general linear ansatz for $F^{\alpha a}$ satisfying (3.6) is then

$$F^{\alpha a} = \epsilon^{ab} \phi_{\beta b}^* \kappa^{\beta\alpha} (-1)^{\varepsilon_\alpha}, \quad (\text{A.11})$$

which when together with $F_\alpha = 0$ are inserted into (A.10) exactly reproduces (A.4). Different matrices $\kappa^{\alpha\beta}$ are related by the gradient shifts (3.11) apart from the absolute normalization. We have

$$F^{\alpha a} \mapsto F^{\alpha a} + \frac{\partial \mathcal{H}}{\partial \phi_{\alpha a}^*}, \quad (\text{A.12})$$

where the most general form of \mathcal{H} is

$$\mathcal{H} = \frac{1}{2} \epsilon^{ab} \phi_{\alpha a}^* \sigma^{\alpha\beta} \phi_{\beta b}^*, \quad (\text{A.13})$$

where $\sigma^{\alpha\beta}$ is a constant symmetric matrix, *i.e.* $\sigma^{\beta\alpha} = \sigma^{\alpha\beta} (-1)^{\varepsilon_\alpha \varepsilon_\beta}$. The shift (A.12) is then equivalent to $\kappa^{\alpha\beta} \mapsto \kappa^{\alpha\beta} + \sigma^{\alpha\beta}$. This generalizes the gradient shifts considered for \mathcal{V}^a in (4.9), (4.21).

The next condition involves ω^{AB} defined by (3.18). In terms of the Darboux coordinates (A.1) ω^{AB} becomes

$$\omega^{AB} = \begin{pmatrix} \omega^{\alpha\beta} & \omega^\alpha_{\beta b} \\ \omega_{\alpha a}^\beta & \omega_{\alpha a \beta b} \end{pmatrix} = \begin{pmatrix} \omega^{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix}, \quad (\text{A.14})$$

where

$$\begin{aligned}\omega^{\alpha\beta} &= \frac{1}{2} \epsilon_{ab} \left(\frac{\partial F^{\beta b}}{\partial \phi_{\alpha a}^*} - \frac{\partial F^{\alpha a}}{\partial \phi_{\beta b}^*} (-1)^{(\varepsilon_\alpha+1)(\varepsilon_\beta+1)} \right) (-1)^{\varepsilon_\beta} = \\ &= \kappa^{\alpha\beta} - \kappa^{\beta\alpha} (-1)^{\varepsilon_\alpha \varepsilon_\beta}.\end{aligned}\quad (\text{A.15})$$

The maximal rank of $\omega^{\alpha\beta}$ is obviously $2N$ since the number of field components are $2N$ according to (A.1). This is also the rank that follows from the V^a of refs.[2]-[5].

The Poisson bracket (3.25) is defined in terms of ω^{AB} and due to (A.14) it is here given by

$$\{F, G\} = F \overleftarrow{\partial}_\alpha \omega^{\alpha\beta} \partial_\beta G, \quad (\text{A.16})$$

which is nondegenerate on the $2N$ dimensional field manifold $\mathcal{L}_1 = \{\phi^\alpha\}$. One may notice that $\kappa^{\alpha\beta}$ in the symmetric V^a of [5] and [1] satisfies $\kappa^{\alpha\beta} = -\kappa^{\beta\alpha}(-1)^{\varepsilon_\alpha\varepsilon_\beta}$ which implies $\omega^{\alpha\beta} = 2\kappa^{\alpha\beta}$ and that this antisymmetric $\kappa^{\alpha\beta}$ cannot be modified by the gradient shift (A.12) since $\sigma^{\alpha\beta}$ in (A.13) is a symmetric matrix.

Finally we give the one-form (3.16) in the Darboux coordinates (A.1). From (A.11) we find

$$\omega_1^a = \epsilon^{cb} \phi_{\beta b}^* \kappa^{\beta\alpha} (-1)^{\varepsilon_\alpha} d^a \phi_{\alpha c}^* + d^a \mathcal{H}, \quad (\text{A.17})$$

where \mathcal{H} satisfies the classical master equation, and (3.17) reduces to

$$d^{\{a} \omega_1^{b\}} = -\frac{1}{2} \epsilon^{cd} \omega^{\alpha\beta} (-1)^{\varepsilon_\alpha + \varepsilon_\beta} d^{\{b} \phi_{\beta d}^* \wedge d^{a\}} \phi_{\alpha c}^*. \quad (\text{A.18})$$

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